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A finiteness condition on subgroups of large derived length

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Abstract

The authors discuss the class $\mathfrak{S}_d(r)$ of groups in which every finitely generated subgroup is either at most r-generated or soluble of derived length at most d. Such groups need not be of finite rank or soluble of derived length at most d in general. A structure theorem is obtained for locally finite, and for certain locally nilpotent, $\mathfrak{S}_d(r)$-groups.

Keywords: Locally finite; Locally nilpotent; Finite rank; Soluble

1. Introduction

A group G is said to have finite (Prüfer) rank r if every finitely generated subgroup of G is at most r-generator, and r is the least natural number with this property. Groups with finite rank have made numerous appearances in the literature (see, for example, [24]). More recently the authors in [8,9] discussed groups in which every proper subgroup is either nilpotent of class at most c (respectively soluble of derived length at most d) or of

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finite rank. In [18] this approach was varied by considering the class $\mathfrak{N}_c(r)$ of groups in which every finitely generated subgroup is either nilpotent of class at most c or at most r-generator. There the authors were able to show that if G is a locally soluble-by-finite group in the class $\mathfrak{N}_c(r)$ then G is either nilpotent of class at most c or of finite rank. Let $\mathfrak{S}_d(r)$ denote the class of groups in which every finitely generated subgroup is either at most r-generated or soluble of derived length at most d. In [8] we showed that a result analogous to [18, Theorem 1] does not hold for the class $\mathfrak{S}_d(r)$ by exhibiting a 3-generator group of infinite rank, soluble of derived length 3, in which every proper subgroup is either 3-generator or of derived length at most 2.

In Section 4 of this paper we give a further example to show that the class $\mathfrak{S}_d(r)$ is not as well behaved as $\mathfrak{N}_c(r)$ by exhibiting a locally nilpotent group of infinite rank in $\mathfrak{S}_d(r)$ that is not soluble of derived length d. In Sections 2 and 4 we show, however, that locally nilpotent groups in the class $\mathfrak{S}_d(r)$ with infinite torsion-free rank are soluble of derived length at most d and that in any case locally nilpotent $\mathfrak{S}_d(r)$-groups either have finite rank or are soluble. We recall that a group is *almost locally soluble* if it has a locally soluble subgroup of finite index. We let $\mathfrak{S}_d$ and $\mathfrak{R}$ denote the classes of groups that are soluble of derived length at most d and of finite rank, respectively. In Section 3 we show that locally finite groups in the class $\mathfrak{S}_d(r)$ are quite well-behaved by proving the following result.

Theorem 1. *Let G be a locally finite group and suppose that every finite subgroup of G is either r-generated or soluble of derived length at most d.*

(i) *If G is almost locally soluble, then either G is soluble of derived length at most d or G has finite rank.*
(ii) *If $G \notin \mathfrak{S}_d \cup \mathfrak{R}$, then there is a normal subgroup M of G such that G/M is soluble of derived length at most d and of rank at most r, and M is isomorphic to one of $PSL(2,K)$, $PSU(3,K)$, ${}^2B_2(K)$, ${}^2G_2(K)$, $SL(2,K)$, or $SU(3,K)$, for some infinite locally finite field K.*

It is clear that the class $\mathfrak{S}_d(r)$ is closed under taking subgroups and quotient groups. The following result, which is easily proved, will also be used throughout this paper.

Lemma 1. *Let $G \in \mathfrak{S}_d(r)$ and suppose that $N \lhd G$.*

(a) *If $N \notin \mathfrak{S}_d$, then G/N has rank at most r.*
(b) *If N is a maximal $\mathfrak{S}_d$-subgroup, then G/N has rank at most r.*

The proof of the following result is entirely analogous to the proof of [18, Proposition 2.1], so we omit it.

Lemma 2. *Suppose that the group G has a normal nilpotent subgroup N of class c and finite index k and suppose that $G \in \mathfrak{S}_d(r)$. Then either G is soluble of derived length at most d or G has finite rank, bounded in terms of c, k, and r.*

Our notation is standard and is that of [23].

2. Torsion-free locally nilpotent groups

We first consider torsion-free locally nilpotent groups in the class $\mathfrak{S}_d(r)$. The following result will prove useful.

Lemma 3. *Let $G = A \rtimes F$ for some finitely generated torsion-free abelian normal subgroup A and some subgroup F. Suppose that F acts nilpotently on A and that, for some positive integer r, every F-submodule of A is r-generated as an F-module. Then $[A,_r F] = 1$.*

Proof. Suppose that the result is false and let $A_0 = A$ and $A_{i+1} = [A_i, F]$, for each $i \geqslant 0$. If A_i/A_{i+1} is periodic for some i then so is A_{i+1}/A_{i+2} and, inductively, A_k/A_{k+1} is periodic for all $k \geqslant i$, which is a contradiction. Thus, if T/A_{r+1} denotes the torsion subgroup of A/A_{r+1}, we may factor by T and hence assume that $A_{r+1} = 1$. Let p be a prime and set $B = \langle A_0^{p^r}, A_1^{p^{r-1}}, \dots, A_i^{p^{r-i}}, \dots, A_{r-1}^p, A_r\rangle$. Since A_i/A_{i+1} is non-periodic for $i = 0, \dots, r$, the 0-rank of A is at least $r + 1$, as therefore is the 0-rank of B. Since B is free abelian, the Prüfer rank of B/B^p has the same value as the 0-rank of B.

Clearly B is an F-submodule of A and, since $[A_i^{p^{r-i}}, F] = A_{i+1}^{p^{r-i}} = (A_{i+1}^{p^{r-(i+1)}})^p$ for $i \geqslant 0$, it follows that $[B, F] \leqslant B^p$, so B/B^p is a trivial F-module. Since B/B^p is not r-generated as an abelian group, it is therefore not r-generated as an F-module, hence neither is B, a contradiction. □

Proposition 1. *Let G be a locally nilpotent group, T the torsion subgroup of G, and suppose that there is a positive integer r and a finitely generated subgroup F of G such that every finitely generated subgroup of G containing F is r-generated. Then G/T is nilpotent of finite rank, and both its rank and class are bounded in terms of r and the rank of F.*

Proof. We may suppose that $T = 1$. It suffices to bound the rank of an arbitrary finitely generated subgroup of G, since a torsion-free nilpotent group of rank t has class at most t. So we may further assume that G is finitely generated. Let A be an arbitrary F-invariant abelian subgroup of G. In order to show that G has rank bounded in terms of r and the rank of F it suffices, by [23, Lemma 6.37], to show that there is a corresponding bound on the rank of A. Thus we may suppose that $G = AF$.

Suppose that F has rank s, and let I be the isolator in A of $F \cap A$. Since G is r-generated, $G = \langle a_1 f_1, a_2 f_2, \dots, a_r f_r\rangle$ for some $a_i \in A$, $f_i \in F$, $i = 1, \dots, r$. Then $G = \langle a_1, \dots, a_r\rangle^F F$ and so A/I is generated as an F-module by the r elements $a_j I$. If V/I is a finitely generated F-invariant subgroup of A/I, then VF is r-generated and hence, as above, V/I is r-generated as an F-module. We deduce from Lemma 3 that $[A/I,_r FI/I] = 1$, that is $[A,_r F] \leqslant I$. Since I has rank at most s, it follows that $[A,_{r+s} F] = 1$ and so G has (r, s)-bounded class c, say, and hence (r, s)-bounded rank. □

The following results can now be obtained from Proposition 1.

Theorem 2. *Let G be a torsion-free locally nilpotent group and suppose that $G \in \mathfrak{S}_d(r)$. Then G is soluble of derived length at most d or of finite rank.*

Proof. If $G \notin \mathfrak{S}_d$, then there is a finitely generated subgroup F such that $F \notin \mathfrak{S}_d$. If $F \leqslant K$ and K is finitely generated, then $K \notin \mathfrak{S}_d$ so K is r-generated and the result is immediate from Proposition 1. □

By using a similar argument, we easily obtain the following generalization of this result.

Theorem 3. *Let G be a locally nilpotent group and suppose that $G \in \mathfrak{S}_d(r)$. If G has infinite torsion-free rank then $G \in \mathfrak{S}_d$.*

3. Locally finite groups

Next we consider locally finite groups.

Proposition 2. *Let G be a locally finite group with $G \in \mathfrak{S}_d(r)$ and let K be the locally soluble radical of G. Then $G \in \mathfrak{S}_d$ or K has finite rank.*

Proof. Suppose the result false and let F be a finite subgroup of G with $F \notin \mathfrak{S}_d$. Since K has infinite rank it follows from [12, Theorem 1] that G has an F-invariant abelian subgroup A of infinite rank, and we may assume that $G = AF$. Suppose that F has order n, and let U be an arbitrary finite F-invariant subgroup of A. Then UF is r-generator and so U is s-generator for some $s = s(r, n)$, by the Reidemeister–Schreier theorem. Since every finite subgroup of A is contained in such a subgroup U, we deduce that A has rank at most s, a contradiction. □

Our next result provides the key to our main theorem.

Proposition 3. *Let G be a locally finite group in the class $\mathfrak{S}_d(r)$, and suppose that $G \notin \mathfrak{S}_d \cup \mathfrak{R}$. Then G has a characteristic subgroup M such that G/M has rank at most r and $M/Z(M)$ is infinite simple. Furthermore, the locally soluble radical K of G has finite rank and is soluble of derived length at most d, and $K \cap M = Z(M)$.*

Proof. Let G be as stated, and let K denote the locally soluble radical of G. By Proposition 2 K has finite rank so G/K has infinite rank and it follows that $K \in \mathfrak{S}_d$.

Let R denote the finite residual of G. If G/B is an arbitrary finite image of G, then B is not locally soluble, by Proposition 2, and so G/B has rank at most r. It follows from [20, Theorem O] that G/R is almost locally soluble, so there is a normal subgroup H of finite index in G such that H/R is locally soluble. Clearly H has no insoluble finite images. Let M denote the intersection of all normal subgroups N of G such that G/N is almost locally soluble. Then M is the locally soluble residual of H (and of R), and so H/M is residually locally soluble and hence locally soluble, so that G/M is almost locally soluble. Since M is certainly not locally soluble, it follows that G/M has rank at most r. Also M is perfect, since H/M' is locally soluble.

As a soluble group of finite rank, K has an ascending G-invariant series with factors that are finite (and abelian), and since R centralizes every such factor so does M, and it follows that $K \cap M$ is contained in the hypercentre of M, and hence equals $Z(M)$, since M is perfect. Note that $M/Z(M)$ has infinite rank since both G/M and $Z(M)$ have finite rank. Suppose that $M/Z(M)$ is not simple, and let $N/Z(M)$ be some proper nontrivial normal subgroup. If N is locally soluble, then it is contained in the locally soluble radical of M and hence in K, a contradiction. So N is not locally soluble, and M/N has rank at most r. If L denotes the intersection of all such subgroups N, then M/L is almost locally soluble, as was the case for G/M. By Proposition 2 and the fact that M is perfect, we deduce that M/L has finite rank. Now a locally soluble group of finite rank is residually soluble [23, Lemma 10.39] and, again because M is perfect, we deduce that the locally soluble radical Q/L of M/L is a proper subgroup of finite index. However, Q is normal in G, and we see (by considering the centralizer of Q/L in G) that G/Q is almost locally soluble, in contradiction to the definition of M. Thus $M/Z(M)$ is simple, as claimed, and the proof of the proposition is complete. □

As Proposition 3 suggests, we now need some information concerning simple locally finite groups in the class $\mathfrak{S}_d(r)$. The following result uses the classification of simple linear locally finite groups as expounded in [2,3,14,28]; such groups are of Lie type over locally finite fields K. Our notation for groups of Lie type generally follows that of [5]; in particular, we adopt the convention that all groups mentioned exist. For instance, it is implicit in our use of the notation ${}^2G_2(K)$ that K has characteristic 3. However, when the field involved is finite we often follow [27] so that $SU(3,q)$ (and *not* $SU(3,q^2)$) denotes the subgroup of all elements in $SL(3,q^2)$ that preserve a fixed non-degenerate Hermitian form on the 3-dimensional vector space over the field of order q^2. This lack of consistency enables us to cite the standard references; our intentions should always be clear from the context.

Proposition 4. *Let r be a fixed positive integer. Let G be an infinite simple locally finite group with all finite subgroups either soluble or at most r-generated. Then G is isomorphic to one of the groups of Lie type $A_1(K)$, ${}^2A_2(K)$, ${}^2B_2(K)$, or ${}^2G_2(K)$ for some infinite locally finite field K.*

Proof. Suppose that G is not linear. Then, by [13, Lemma 3.24], there exists a subgroup N of G and a family of finite subgroups $\{N_i\}_{i\geqslant 1}$ of N such that $N = \langle N_i \mid i \geqslant 1\rangle$, $N_i \cap N_{i+1}N_{i+2}\cdots = 1$, N_i normalizes N_j for $i \leqslant j$ and the product n_i of the orders of the non-abelian composition factors of N_i has the property that $n_i \to \infty$ as $i \to \infty$. Evidently N is not locally soluble and so a result of Sunkov [29] shows that N contains an abelian subgroup of infinite rank. Let A be a finite abelian subgroup of N that cannot be generated by r elements and let k be such that $A \leqslant \langle N_1, N_2, \ldots, N_k\rangle$. Let $j > k$ be such that N_j is insoluble and note that A normalizes N_j. Clearly N_jA is finite, insoluble, and cannot be generated by r elements. From this contradiction we deduce that G is linear and so G is a group of Lie type defined over an infinite locally finite field K, of characteristic p, say.

Suppose G is not of one of the types referred to in the theorem. We derive a contradiction in the following way: we find subgroups M and N of G such that M is not locally soluble, N is a nilpotent p-group of infinite (Prüfer) rank and M normalizes N. With such subgroups M and N in hand, let M^* be a finite insoluble subgroup of M. Then

$NM^* \in \mathfrak{S}_d(r)$, for some integer d, dependent upon M^* and by Proposition 2 we obtain the contradiction that NM^* is soluble of derived length at most d.

We proceed to find such subgroups M and N of G. Suppose first that G is untwisted, let Π denote a system of fundamental roots and let Φ^+ denote the corresponding system of positive roots (thus Π may be viewed as the set of nodes of the Dynkin diagram associated to G). Let $s \in \Pi$, $M = \langle X_s(K), X_{-s}(K)\rangle$ and $N = \langle X_t(K) \mid t \in \Phi^+, t \neq s\rangle$. Now M is not locally soluble since it is a non-trivial image of $SL(2, K)$ and clearly N is a nilpotent p-group of infinite rank. Moreover, the Chevalley commutator formula shows that M normalizes N as required.

We next consider twisted groups of Lie type. We shall use the notation of [5] throughout and so it is appropriate to denote the twisted group under consideration by G^1 instead of G. In this way we may consider G^1 as a twisted version of a Chevalley group G in accordance with [5]. Let Π be a fundamental system of roots for the (untwisted) group G and let Φ^+ denote the corresponding system of positive roots. Let the equivalence classes of Φ^+, in the sense of [5, 13.2], be $S_1, S_2, \dots, S_c$. Note that our assumptions about G^1 imply that $c \neq 1$. Moreover, there is no loss in assuming that either $S_1 = \{s\}$ for some fundamental root s or $S = \{s, \bar{s}\}$ for some fundamental root s such that $s + \bar{s}$ is not a root. Let $M = \langle X^1_{S_1}, X^1_{-S_1}\rangle$ and $N = \langle X^1_{S_i} \mid i = 2, 3, \dots, c\rangle$ (here, of course, $-S_1 = \{-v \mid v \in S_1\}$). That M is not locally soluble follows from [25, p. 183] while N is easily seen to be a nilpotent p-group of infinite rank. Moreover, Chevalley's commutator formula again shows that M normalizes N. The result now follows. $\square$

Proposition 5. *Let G be isomorphic to one of the groups of Lie type $A_1(K)$, ${}^2A_2(K)$, ${}^2B_2(K)$, or ${}^2G_2(K)$ for some infinite locally finite field K. Then $G \in \mathfrak{S}_6(4)$.*

Proof. By the classification of simple linear locally finite groups (see [2,3,14,28]), each finite subgroup of $A_1(K)$ lies in a subgroup of the form $A_1(F)$ for some *finite* subfield F of K; similar remarks apply to the groups of types ${}^2A_2$, ${}^2B_2$, and ${}^2G_2$. Thus the proposition will follow once we show that there exist integers $d_1, d_2, d_3, d_4, r_1, r_2, r_3, r_4$ such that for each finite field F we have that $A_1(F) \in \mathfrak{S}_{d_1}(r_1)$, ${}^2A_2(F) \in \mathfrak{S}_{d_2}(r_2)$, ${}^2B_2(F) \in \mathfrak{S}_{d_3}(r_3)$, ${}^2G_2(F) \in \mathfrak{S}_{d_4}(r_4)$ where $2 \leqslant d_i \leqslant 6$, $2 \leqslant r_i \leqslant 4$ for $i = 1, 2, 3, 4$.

Although we do not require the full strength of the result, it is useful at this point to note that every finite simple group is 2-generator [1, Theorem B].

A complete list of the isomorphism types of subgroups of $A_1(F) \cong PSL(2, F)$ is given in [27, Theorem 6.25]. The soluble subgroups have derived length at most 3 and the insoluble subgroups are either simple (and hence 2-generator) or of the form $PGL(2, q)$, where q is a power of p. Since an insoluble group $PGL(2, q)$ is an extension of its simple socle by a cyclic group, a result of Lucchini [19, Theorem] shows that $PGL(2, q)$ is 2-generator if it is insoluble. It follows that $A_1(K) \in \mathfrak{S}_3(2)$.

The group ${}^2B_2(F)$ is a Suzuki group. Its subgroups are given in [26, p. 138]; each is either metabelian or (simple and) 2-generator. It follows that ${}^2B_2(K) \in \mathfrak{S}_2(2)$.

The group ${}^2G_2(F)$ is a Ree group. Its subgroups can be found in [16,17,30]; each is either soluble of derived length at most 3 or of the form $PSL(2, D)$, ${}^2G_2(D)$ or $\langle\eta\rangle \times PSL(2, D)$, for some subfield D of F and some involution η. Since $PSL(2, D)$ can be generated by two elements one of which has odd order (see, for instance, [4]), it follows easily that ${}^2G_2(K) \in \mathfrak{S}_3(2)$.

Finally we consider the group ${}^2A_2(F)$ and claim that this belongs to $\mathfrak{S}_6(4)$. Here, by the convention stated before Proposition 4, $|F| = q^2$ for some prime power q. Let V be a 3-dimensional vector space over F equipped with a non-degenerate Hermitian form $\langle\, ,\, \rangle$. We view $SU(3,q)$ as the subgroup of $SL(3,q^2)$ that preserves $\langle\, ,\, \rangle$ in the usual way. Hence, as is well known, ${}^2A_2(F) \cong PSU(3,q)$ where $PSU(3,q)$ denotes the image of $SU(3,q)$ under the natural map $SL(3,q^2) \to PSL(3,q^2)$.

We begin by observing that it follows easily from [21, Theorems 2.11 and 2.12] that soluble subgroups of $GL(2,q^2)$ and $GL(3,q^2)$ have derived lengths at most 4 and 6 respectively. Consequently soluble subgroups of $PSU(3,q)$ have derived length at most 6. The maximal subgroups of $PSU(3,q)$ were determined in [15,22]; a modern treatment can be found in [11, Theorem 6.5.3] and it is to this we refer throughout. In the proof of that theorem it is shown that a subgroup H of $PSU(3,q)$ is of one of the following three types:

(i) the image under the natural map from $SU(3,q)$ of a subgroup $\widehat{H}$ that acts reducibly on V;
(ii) soluble (and therefore of derived length at most 6);
(iii) the socle H_0 of H is simple and isomorphic to a simple group of the form $PSL(2,Q), PSL(3,Q), PSU(3,Q)$ for some field Q, or Alt(7), the alternating group of degree 7.

(Note that Alt(6) $\cong PSL(2,9)$ and that the socle of the Mathieu group M_{10} is isomorphic to $PSL(2,9)$.) The outer automorphism group of H_0 is therefore cyclic or metacyclic (see, for instance, [6, Table 5]) and it follows that H is at most 4-generator in this case. In fact, it follows from [19] that such an H is 2-generator. Consequently, to establish our claim, and therefore complete the proof of the proposition, it suffices to show that each insoluble group H of type (i) above is 4-generator.

To this end, let $\widehat{H} \leqslant SU(3,q) \leqslant SL(3,q^2)$ and $H \leqslant PSU(3,q)$ be insoluble groups of the kind referred to in type (i) above. Suppose that W is a minimal $\widehat{H}$-invariant non-zero subspace of V. If W is degenerate, 1-dimensional and $0 \neq w \in W$, the kernel V_1 of the linear map $\langle\, , w\rangle : V \to F$ is a proper $\widehat{H}$-invariant subspace of V that contains W. Thus $0 \precneqq W \precneqq V_1 \precneqq V$ is a series of $\widehat{H}$-invariant subspaces of V and it is easy to see that $\widehat{H}$ is an extension of the nilpotent group $C_{\widehat{H}}(W) \cap C_{\widehat{H}}(V_1/W) \cap C_{\widehat{H}}(V/V_1)$ by an abelian group, and so H is soluble, a contradiction. If W is degenerate and of dimension 2, we deduce that $W = W^\perp$ which yields a contradiction since $\dim(V_0) + \dim(V_0^\perp) = 3$ for all subspaces V_0 of V. Therefore, we may assume that W is non-degenerate and it follows that V is the direct sum of $\widehat{H}$-invariant subspaces $V = W \oplus W^\perp$. As above, the insolubility of $\widehat{H}$ implies that W and $W^\perp$ are minimal $\widehat{H}$-invariant subspaces of V and so there is no loss in assuming that W has dimension 2. Let $\widehat{N}$ be the normal subgroup of $\widehat{H}$ consisting of all elements that act trivially on $W^\perp$. Since elements of $\widehat{H}$ have determinant 1 it follows easily that $\widehat{H}/\widehat{N}$ is cyclic. Now the insoluble group $\widehat{N}$ is isomorphic to a subgroup of $SU(2,q) \cong SL(2,q)$, (see, for instance, [27, p. 383] for information about this isomorphism), and so $\widehat{N}$ is an extension of a cyclic central subgroup by an insoluble subgroup of $PSL(2,q)$. As we have seen above, insoluble subgroups of $PSL(2,q)$ are 2-generator and we deduce that $\widehat{N}$ is

3-generator, $\widehat{H}$ is 4-generator and H is 4-generator. Thus each group H of type (i) above belongs to $\mathfrak{S}_6(4)$ and the proof is complete. $\square$

We come now to the proof of our main theorem.

Proof of Theorem 1. Part (i) follows immediately from Proposition 2. To prove (ii) we note that Proposition 3 implies that G contains a normal subgroup M such that G/M has rank at most r and $M/Z(M)$ is an infinite simple group. Hence $M/Z(M)$ is one of the simple groups listed in Proposition 4. Using the argument of [7, Lemma 3.4] and the fact that the Schur multiplier of $PSU(3,K)$ is either trivial or cyclic of order 3 (and this depends upon the subfields of K) it follows that M is one of the groups listed in the theorem. Let p be the characteristic of the underlying field K of the linear group M. The Sylow p-subgroups of M are of infinite rank and conjugate (by, for example, [31, 9.10 Theorem]), and the Frattini argument gives $G = N_G(U)M$. Since $U \lhd N_G(U)$, Proposition 2 implies that $N_G(U) \in \mathfrak{S}_d$, and hence $G/M \in \mathfrak{S}_d$. This completes the proof. $\square$

We note the following converse to Theorem 1.

Theorem 4. *Let G be a locally finite group containing a normal subgroup M isomorphic to one of $PSL(2,K)$, $PSU(3,K)$, ${}^2B_2(K)$, ${}^2G_2(K)$, $SL(2,K)$, or $SU(3,K)$, for some locally finite field K and suppose that G/M is soluble of derived length at most e and of finite rank at most s. Then $G \in \mathfrak{S}_d(r)$ for some d, r, depending only on e and s.*

Proof. Let G be as stated and let F be a finite subgroup of G. Then $F/F \cap M \cong FM/M$ is soluble of derived length at most e and of course is at most s-generator. Now if $F \cap M$ is a soluble subgroup of M then it is of derived length at most 6, so that F is soluble of derived length at most $e+6$. If $F \cap M$ is not soluble then it is at most 5-generator and hence F is at most $(s+5)$-generator. Thus we may take $d = e+6$ and $r = s+5$. The result follows. $\square$

4. Examples and a further result

In this final section we give the examples promised in Section 1 and one further result which should be compared to Theorems 2 and 3.

Theorem 5. *For each $d \geqslant 2$ there exists a locally nilpotent group G of infinite rank that is not soluble of derived length d such that $G \in \mathfrak{S}_d(r)$.*

Proof. We present here the proof in the case when $d = 2$. The general case is almost identical and the reader is referred to the proof of [10, Theorem 5].

Let p be a fixed prime and (with some abuse of notation) let $F = \langle x, y_1, y_2 \mid R\rangle$ where R is the set of relations $\{[y_1, y_2] = 1, \gamma_5(F) = 1, (F'')^p = 1\}$. Then $F'' \neq 1$ and $H := \langle F', y_1, y_2, x^p\rangle \in \mathfrak{S}_2$. Let $C = \langle F', y_1, y_2\rangle$. Set $A = \langle a_1\rangle \times \langle a_2\rangle \times \cdots$ where each $|a_i| = p$

and define an action of F on A via $[A, C] = 1$, $[a_{i+1}, x] = a_i$ for all $i \geqslant 1$ and $[a_1, x] = 1$. Let $G = A \rtimes F$. Clearly G is locally nilpotent, of infinite rank, and $G'' \neq 1$ (since $F'' \neq 1$).

Let K be a finitely generated subgroup of G and suppose that K is not metabelian. Then $K \nleqslant AH$ since $(AH)' = A'H'[A, H] = [A, H]H'$ and $(AH)'' = [A, H]'H''[A, H, H'] = 1$, since $H'' = 1$ and $H' \leqslant C$, which centralizes A. Thus K contains an element of the form $x^r ca$, where $c \in C$, $a \in A$ and p does not divide r (since $x^p \in H$). Choose r least such, then $K = (K \cap AH)\langle x^r ca\rangle$.

Now $A \cap K$ is normalized by K and by AC, so $A \cap K$ is normalized by $\langle x^r\rangle$ and hence by $\langle x\rangle$ (this follows using isolators and the fact that $A \cap K$ is a p-group, and $(p, r) = 1$).

It is easy to see that the only finite $\langle x\rangle$-submodules (indeed, the only proper submodules) of A are 1 and the subgroups $\langle a_1, \dots, a_t\rangle$ for each positive integer t. If $A \cap K = 1$ then K is isomorphic to a subgroup of G/A and hence to a subgroup of F, and if r is the rank of F (which is finite, since F is finitely generated nilpotent) then K is at most r-generator. So assume that $K \cap A = \langle a_1, \dots, a_t\rangle$. Then $A \cap K = \langle a_t\rangle^{\langle x\rangle}$ and hence $A \cap K = \langle a_t\rangle^{\langle x_0^r\rangle}$, since $\langle a_t\rangle^{\langle x^r\rangle}$ is normalized by $\langle x^r\rangle$ and hence by $\langle x\rangle$. Thus $A \cap K = \langle a_t\rangle^K$ and K is $(r + 1)$-generator in this case. $\square$

Theorem 6. *Let G be a locally nilpotent group with $G \in \mathfrak{S}_d(r)$. Then either G has finite rank or G is soluble.*

Proof. Let T denote the torsion subgroup of G. If T is not soluble then T has finite rank, by Theorem 1, and G/T has rank at most r, by Lemma 1(a) so that G has finite rank. On the other hand, G/T is in any case soluble, by Theorem 2 and the fact that a torsion-free locally nilpotent group of finite rank is nilpotent, so if T is soluble then so is G and the result follows. $\square$

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